The Ratio (Orlov-Kindem) Method

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1 Abstract

The ratio method was conceived by Yuri Orlov and technically implemented by Joel Kindem in his analysis of the 1997 data [1]. This note will describe the ratio formulation and add a refinement to Joel’s original implementation. It will be shown that at the statistical level of the 1999 data analysis (∼ 1.2 ppm uncertainty in $\omega_h$), the distribution of $(\chi^2/ndf)$ for the ratio fitting procedure is Gaussian with mean $(1,00069 \pm 0.00047)$ and width $(0.96 \pm 0.01)(\sqrt{2/ndf})$, and that the systematic uncertainty in fitting for $\omega_a$ is 0.05 ppm. Due to the time shifting in the ratio method, there is an inevitable statistical fluctuation when comparing the fitted values of $\omega_h$ between the ratio and five-parameter fits. It will be shown that the distribution of values $\mathcal{D} = \omega_h$ (five-parameter) $- \omega_a$ (ratio) is a Gaussian with mean zero and width $\approx 13\%$ of the statistical uncertainty (e.g. 1.2 ppm statistical uncertainty $\rightarrow 0.16$ ppm fluctuation between the ratio and five-parameter fits). The note will end with the derivation of a mathematical formula for estimating systematic errors in any fitting procedure due to neglecting background terms. The formula will then be applied to the specific case of the ratio fitting procedure.

2 The Ratio Formulation

Assume that the observed time spectrum $n(t)$ is given by the normal five-parameter function plus an additive background term $b(t)$

$$n(t) = N_0 e^{-\tau t} (1 - A \cos(\omega t + \phi)) + b(t)$$

(1)
Let $\tau_a$ be the anomalous precession period and define functions $u(t)$ and $v(t)$ as

$$u(t) = n \left( t + \frac{\tau_a}{2} \right) + n \left( t - \frac{\tau_a}{2} \right) = n^+ + n^- \quad (2)$$

$$v(t) = 2n(t) = 2n^0 \quad (3)$$

Then $R(t)$ is given by the formula

$$R(t) = \frac{u(t) - v(t)}{u(t) + v(t)} = \frac{n^+ + n^- - 2n^0}{n^+ + n^- + 2n^0} \quad (4)$$

The value of $\tau_a$ is known with some uncertainty $\delta$ so that in Eq. 2, $\tau_a \rightarrow (\tau_a + \delta)$. Expanding the exponential to second order in $(\tau_a/\tau_\mu)$ and the cosine to first order in $(\delta/\tau_a)$ gives

$$R(t) = \left[ 1 - \epsilon_1 - \epsilon_2 - b_{\text{den}} \right] \ast A_0 \cos(\omega t + \phi) + \epsilon_1 + \epsilon_2 + b_{\text{num}} \quad (5)$$

where

$$\epsilon_1 = \left( \frac{\delta}{\tau_\mu} \right) \pi A_0 \sin(\omega t + \phi) \quad (6)$$

$$\epsilon_2 = \frac{1}{16} \left( \frac{\tau_a}{\tau_\mu} \right)^2 (1 + A_0 \cos(\omega t + \phi)) \quad (7)$$

$$b_{\text{num}} = \frac{b^+ + b^- - 2b^0}{4N_0 e^{\frac{-\lambda}{\tau}} } \quad (8)$$

$$b_{\text{den}} = \frac{b^+ + b^- + 2b^0}{4N_0 e^{\frac{-\lambda}{\tau}} } \quad (9)$$

and the functions $(b^+, b^-, b^0)$ are defined analogously as $(n^+, n^-, n^0)$ in Eqs. 2 – 3. We certainly know the value of $\tau_a$ to $< 10$ ppm, and hence $\epsilon_1$ has magnitude $< 10^{-5} \times (\tau_a/\tau_\mu) \approx 7 \times 10^{-7}$ which is negligible ($\pi A_0 \sim 1$). Substituting the expression for $\epsilon_2$ in Eq. 7 into Eq. 5 gives

$$R(t) = \left[ 1 - b_{\text{den}} \right] \ast A_0 \cos(\omega t + \phi) + b_{\text{num}} + \epsilon - \epsilon(t) \quad (10)$$

where
\[ \epsilon = \frac{1}{16} \left( \frac{\tau_{\text{e}}}{\tau_{\mu}} \right)^2 = \frac{1}{16} \left( \frac{4.3654}{64.38} \right)^2 = 0.00028736 \]  
(11)  
\[ \lambda(t) = A_0 \cos^2(\omega t + \phi) \]  
(12)  
It will be shown in section 5 that the term \( \epsilon \lambda(t) \) contributes < 0.01 ppm to the systematic uncertainty in fitting for \( \omega \) and hence can be neglected. Therefore, the final functional for the ratio fitting procedure is

\[ R(t) = [1 - b_{\text{den}}] \ast A_0 \cos(\omega t + \phi) + \epsilon + b_{\text{num}} \]  
(13)  
where \( \epsilon \) is as given in Eq. 11. The minor refinement of the formulation in [1] is the addition of the additive constant term \( \epsilon \). It will be shown in section 5 that the absence of this term in the fit will cause an error in the fitted value of \( \omega \) which oscillates at frequency \( \omega \) and has magnitude \( \approx 0.2 \) ppm. Therefore, the \( \epsilon \) term must be retained in Eq. 13.

Assuming that pileup effects contribute a background term \( b(t) \) as follows

\[ b(t) = \epsilon_p N_0 e^{\frac{t}{\tau_0}} (1 - A_p \cos(\omega t + \phi)) \]  
(14)  
where \( \epsilon_p \) is the fraction of the number of doubles to singles at injection, and \( A_p \) is the doubles asymmetry. Note that this description of the doubles time spectrum neglects the phase difference between the singles and doubles terms as well as the double frequency component. Eqs. 8 and 9 then give, neglecting higher order terms,

\[ b_{\text{num}}(t) = \epsilon_p e^{\frac{t}{\tau_0}} A_p \cos(\omega t + \phi) \]  
(15)  
\[ b_{\text{den}}(t) = \epsilon_p e^{\frac{t}{\tau_0}} \]  
(16)  
and the ratio functional takes the form

\[ R(t) = [1 - K e^{\frac{t}{\tau_0}}] \ast A_0 \cos(\omega t + \phi) + \epsilon \]  
(17)  
where

\[ K = \epsilon_p \left( \frac{A_0 - A_p}{A_0} \right) \]  
(18)
3 Implementation of the Ratio Fitting Procedure

3.1 Error Propagation

For Poisson statistics (counting of positrons) the variance in a time bin \( i \) is given by \( \sigma_n^2 = n(i) \). The variance \( \sigma_R^2 \) in \( R(i) \) for the same bin \( i \) is given via the error propagation formula

\[
\sigma_R^2 = \left( \frac{\partial R}{\partial u} \right)^2 \sigma_u^2 + \left( \frac{\partial R}{\partial v} \right)^2 \sigma_v^2. \tag{19}
\]

There are two ways to propagate errors for \( v \). In the first \( v = n^o + n^\circ \) specifying that there will be two histograms filled using two distinct sets of decay positrons, and in the second \( v = 2n^\circ \) specifying that there will be one histogram filled using one distinct set of decay positrons with the histogram values multiplied by 2. In the first case the variances \( \sigma_u^2, \sigma_v^2, \) and \( \sigma_R^2 \) are

\[
\sigma_u^2 = \left( \frac{\partial u}{\partial n^+} \right)^2 \sigma_{n^+}^2 + \left( \frac{\partial u}{\partial n^-} \right)^2 \sigma_{n^-}^2 = \sigma_{n^+}^2 + \sigma_{n^-}^2 = n^+ + n^- = u \tag{20}
\]

\[
\sigma_v^2 = \left( \frac{\partial v}{\partial n^\circ} \right)^2 \sigma_{n^\circ}^2 + \left( \frac{\partial v}{\partial n^\circ} \right)^2 \sigma_{n^\circ}^2 = \sigma_{n^\circ}^2 + \sigma_{n^\circ}^2 = n^\circ + n^\circ = v \tag{21}
\]

\[
\sigma_R^2 = \frac{4uv}{(u+v)^3} = \frac{1 - R^2}{u+v} \tag{22}
\]

and in the second case

\[
\sigma_u^2 = \left( \frac{\partial u}{\partial n^+} \right)^2 \sigma_{n^+}^2 + \left( \frac{\partial u}{\partial n^-} \right)^2 \sigma_{n^-}^2 = \sigma_{n^+}^2 + \sigma_{n^-}^2 = n^+ + n^- = u \tag{23}
\]

\[
\sigma_v^2 = \left( \frac{\partial v}{\partial n^\circ} \right)^2 \sigma_{n^\circ}^2 = 4\sigma_{n^\circ}^2 = 2v. \tag{24}
\]

\[
\sigma_R^2 = \frac{4uv(2u+v)}{(u+v)^4} = \frac{(1 - R^2)2u+v}{(u+v)^2}. \tag{25}
\]

The simpler form of the propagated error in Eq. 22 is preferred when using the ratio formulation, and hence four distinct sets of positron data are used (one for \( n^+ \), one for \( n^- \), and two for \( n^\circ \) and \( n^\circ \)).
3.2 Fitting

For extracting parameters from experimental measurements, the principle of maximum likelihood is usually invoked. The likelihood \( \mathcal{L}(p_1, \ldots, p_m) \) for observing a histogrammed data set \( (n_i : i = 1, \ldots, N) \) is a product of the probabilities \( P_i \) for accumulating the number of counts \( n_i \) in each bin \( i \)

\[
\mathcal{L}(p_1, \ldots, p_m) = \prod_{i=1}^{N} P_i
\]

(26)

For Poisson statistics

\[
P_i = \frac{\mu_i}{n_i!} e^{-\mu_i}
\]

(27)

and for Gaussian statistics

\[
P_i = \frac{1}{\sqrt{2\pi\mu_i}} e^{-\frac{1}{2}(\frac{n_i - \mu_i}{\mu_i})^2}
\]

(28)

where \( \mu_i(p_1, \ldots, p_m) \) is the expected count for bin \( i \) given the parent distribution \( D(p_1, \ldots, p_m) \). Assume that counting positrons, which follows Poisson statistics, can be approximated by Gaussian statistics (a good approximation provided \( n_i > 20 \)) and substitute the expression for \( P_i \) in Eq. 28 into Eq. 26. Then the method of maximum likelihood, maximizing \( \mathcal{L} \), is equivalent the method of \( \chi^2 \) minimization, or minimizing the \( \chi^2 \) sum defined by

\[
\chi^2 = \sum_{i=1}^{N} \frac{(n_i - \mu_i)^2}{\mu_i} = \sum_{i=1}^{N} \frac{(n_i - n(t_i))^2}{\sigma_i^2}
\]

(29)

For the ratio fitting procedure, combining Eqs. 22 and 29 gives

\[
\chi^2 = \sum_{i=1}^{N} (u_i + v_i) \frac{(R_i - R(t_i))^2}{1 - R^2(t_i)}
\]

(30)

where the values \( (u_i + v_i) \) and \( R_i \) are gotten from histograms filled using the definitions in Eqs 2 – 4, and the fitting functional \( R(t) \) is given by Eq. 13. Note from the definitions of \( u \) and \( v \) that \( (u_i + v_i) \approx 4n_i \), and hence the condition for Gaussian statistics is \( (u_i + v_i) > 80 \).
4 Test of the Ratio Fitting Procedure

4.1 Simulated Data

A pseudo-Monte Carlo method as outlined in what follows was used to simulate data for studying systematic uncertainties in the ratio fitting procedure. Begin with the five-parameter function \( n(t) \) as in Eq. 1 and set \( \mu(t) = 0 \). For a given time bin \( i \) centered at \( t_i \) with width \( \delta t \), the mean value of \( \mu_i \) is calculated by integrating \( n(t) \) from \( (t_i - \delta t/2) \) to \( (t_i + \delta t/2) \); an exact integration formula was used. Next assume that the observed distribution for that bin is Gaussian with variance \( \sigma_i^2 = \mu_i \). Then use the acceptance/rejection method with a Gaussian function up to 4.5 sigmas to generate a statistical spread \( \delta \mu_i \). The histograms are filled with \( n_i = \mu_i + \delta \mu_i \). To implement the ratio fitting procedure, four sub-histograms of simulated data were filled using the input value \( \tau_o = 4366 \text{ ns} \), and a 10 ppm error was added to the half period shifting \( \pm 2183.02183 \text{ ns} \). A total of 1600 independent sequences of Ranlux pseudo-random numbers were used with the following input parameters for \( n(t) \)

\[
N_o = \left( \frac{2 \times 10^9}{64400} \right) \\
\tau_o = 64400 \text{ ns} \\
A_o = 0.36 \\
\omega_o = \frac{2\pi}{4366 \text{ ns}} \\
\phi = 0
\]

4.2 Fitting Tests

Figure 1 shows the results of ratio fits to 1600 distinct sets of simulated data. The fits were executed for the time range \( 25 - 600 \mu s \) corresponding to bin numbers 168 - 4022 for time bins of width 149.185 \text{ ns} \). In figure 1(a), the mean of the distribution of \( \mathcal{R} = (\omega_{FIT} - \omega_{MC})/\omega_{MC} \) is \( -0.020 \pm 0.030 \text{ ppm} \), and the width is 1.141 \( \pm 0.024 \text{ ppm} \). The expected uncertainty of the fit is then 1.141 ppm. Figure 1(b) shows a plot of the fit uncertainties as returned by MINUIT for the fits in figure 1(a). The mean of that distribution is 1.154 ppm and is consistent with the expected value of 1.141 \( \pm 0.024 \text{ ppm} \). Note that the distribution of the fit uncertainties in figure 1(b) is not Gaussian.
due to the propagated error for the ratio fitting procedure. However, this
effect is negligible because the RMS of the distribution of fit uncertainties
is small, 0.0004 ppm, compared to the mean value of 1.154 ppm. Figure
2 shows a plot of the distribution of $\chi^2/ndf$ for the fit results shown in
figure 1. The distribution is Gaussian with mean $1.00069 \pm 0.00047$ and
width $0.0218 \pm 0.0003$, which is 0.956 times the expected value of $\sqrt{2/ndf} =
\sqrt{2/3852} = 0.0228$.

Figures 3 and 4 show the mean values of $\mathcal{R}$ and $\chi^2/ndf$, respectively,
as a function of fit start times over the range $25 - 140 \mu s$. In figure 3 the
$\mathcal{R}$ values are consistent within the correlated errors to the value at 25 \mu s,
and in figure 4 the $\chi^2/ndf$ versus fit start times is constant at $\approx 1.0007$. In
section 5, it will be shown that the systematic fitting error using the ratio
functional in Eq. 13 is of order 0.01 ppm. However, at the statistical level
of 1.2 ppm, it is sufficient to use the current results to assign a value of 0.05
ppm systematic uncertainty in $\omega_n$ for the ratio fitting procedure. Finally,
the formulation as developed has for its goodness of the fit measure a $\chi^2/ndf$
centered at $\approx 1.0007$.

For completeness the distribution, at 25 \mu s fit start time, of fitted asymmetry
$A_o$ and phases $\phi$ are shown in figures 5 and 6, respectively. Note
that the mean value of $A_o$ is $\approx 0.3553$, compared to the input value of 0.36,
and the mean value of $\phi$ is $-0.049 \pm 0.004$ mrad, compared to the input
value of zero. Therefore, the systematic uncertainty in the fitted value of $\phi$
for the ratio method is $\approx 0.05$ mrad.

4.3 Comparison of the Ratio Fit Results to the Five-Parameter
Fit Results

Figure 7 shows a plot of the distribution of the difference

$$D = \mathcal{R}(\text{five-parameter fit}) - \mathcal{R}(\text{ratio fit})$$

for 1600 distinct sets of simulated data. Time bins of width 136.4375 \text{ ns} were
used to give an even number of time bins (32) per anomalous precession pe-
riod (4366 \text{ ns}). This was necessary to ensure that the generated histograms
for the ratio fitting procedure can be reshifted in time and summed to fill
histograms for the five-parameter fits. The mean of the $D$ distribution is
$0.0075 \pm 0.0067$ ppm, consistent with zero, and the width is $0.143 \pm 0.004$
ppm. The distributions of $\mathcal{R}$ (five-parameter fit) and $\mathcal{R}$ (ratio fit) are shown
in figures 8 and 9, respectively. Note that the two $\mathcal{R}$ distributions are con-
sistent with one another, $\mathcal{R}$ (five-parameter fit) = $0.036 \pm 0.028$ ppm, and
\[ \mathcal{R} \text{(ratio fit)} = 0.044 \pm 0.025 \text{ ppm, and both are consistent with a fitting systematic uncertainty of 0.05 ppm. Note further that the distribution of fit uncertainties for the normal five-parameter fit is Gaussian, as expected, with a width of 0.0002 ppm. Contrastly, the distribution of fit uncertainties for the ratio fit is not Gaussian, due to the propagated error, with a wider width of 0.0004 ppm. Dividing the width of } \mathcal{D}, 0.143 \text{ ppm, by the statistical uncertainty of the five-parameter fit, 1.10 ppm, gives that in a direct comparison between the five-parameter and ratio fits, there is an expected statistical fluctuation of } 0.143/1.10 = 13\% \text{ of the fit uncertainty.} \]

5 Systematic Fitting Errors

5.1 General Formulation

Consider the general case of fitting to data points \( n_i \) the function \( n(i) = y(i) + \epsilon f(i) \) with some variance \( \sigma^2(i) = v(i) = u(i) + \beta w(i) \) where \( \epsilon, \beta \ll 1 \), and assume that all functions depend on the two parameters \( (a, b) \). \( \chi^2 \) minimization leads to

\[
\chi^2(a, b) = \sum \frac{(n_i - n(i))^2}{v(i)} \tag{31}
\]

\[
0 = \frac{\partial \chi^2}{\partial a} = \sum \left( \frac{n_i - n(i)}{v(i)} \right) \left( -2 \frac{\partial n(i)}{\partial a} - \frac{\partial v(i)}{\partial a} \frac{n_i - n(i)}{v(i)} \right)_{a, b} \tag{32}
\]

\[
0 = \frac{\partial \chi^2}{\partial b} = \sum \left( \frac{n_i - n(i)}{v(i)} \right) \left( -2 \frac{\partial n(i)}{\partial b} - \frac{\partial v(i)}{\partial b} \frac{n_i - n(i)}{v(i)} \right)_{a, b} \tag{33}
\]

For an order of magnitude estimate, consider the two terms in the second set of parentheses in Eqs. 32 and 33:

\[
A = \frac{\partial v(i)}{\partial a} \left( \frac{n_i - n(i)}{v(i)} \right) \sim \mathcal{O} \left( \frac{\delta v}{\delta a} \right)
\]

\[
B = \frac{\partial n(i)}{\partial a} \sim \mathcal{O} \left( \frac{\delta n}{\delta a} \right)
\]

\[
\frac{A}{B} \sim \mathcal{O} \left( \frac{\delta v}{\delta n} \right) \sim \mathcal{O} \left( \frac{v}{n} \right)
\]

For the cases of the five-parameter and ratio functionals
\[
\frac{A}{B}^{(\text{five - parameter})} \sim O\left(\frac{\sqrt{n}}{n}\right) \sim \left(\frac{1}{\sqrt{n}}\right) \ll 1
\]

\[
\frac{A}{B}^{(\text{ratio})} \sim O\left(\frac{1 - R^2}{n} \times \frac{1}{R}\right) \sim O\left(\frac{1}{n}\right) \ll 1
\]

Therefore, Eq. 32 can be rewritten as

\[
0 = \frac{\partial^2 \chi^2}{\partial \alpha} \approx -2 \sum \left(\frac{n_i - n(i)}{u(i)}\right) \left(\frac{\partial n(i)}{\partial \alpha}\right)_{a,b}
\]

(34)

Assuming that in the fitting procedure, \(f(i)\) and \(w(i)\) are neglected leading to errors \((\delta a, \delta b)\), then Eq. 34 becomes

\[
0 = \frac{\partial^2 \chi^2}{\partial \alpha} \approx -2 \sum \left(\frac{n_i - y(i)}{u(i)}\right) \left(\frac{\partial y(i)}{\partial \alpha}\right)_{a+\delta a, b+\delta b}
\]

(35)

Now expand Eq. 34 in \((\epsilon, \beta)\) and Eq. 35 in \((\delta a, \delta b)\) to give the equations

\[
0 \approx \sum \left(\frac{n_i - y(i)}{u(i)} - \epsilon \frac{f(i)}{u(i)}\right) \left(1 - \beta \frac{w(i)}{u(i)}\right) \left(\frac{\partial y(i)}{\partial \alpha}\right)_{a,b}
\]

(36)

\[
0 \approx \sum \left(\frac{n_i - y(i)}{u(i)} - \delta \frac{\partial y(i)}{\partial u(i)} - \delta \frac{\partial y(i)}{\partial \alpha} \right) \times \left(\frac{\partial y(i)}{\partial \alpha} + \delta \frac{\partial^2 y(i)}{\partial \alpha \partial \alpha} + \delta \frac{\partial^2 y(i)}{\partial \alpha \partial \beta} \right)_{a,b}
\]

(37)

Keeping only terms of \(O(\epsilon, \delta a, \delta b)\), as well as neglecting the products of those terms with \((n_i - y(i))\) since their sums contain positive as well as negative terms and add to \(\approx 0\), lead to the following approximate equations

\[
0 \approx \epsilon \sum \frac{1}{u(i)} \frac{\partial y(i)}{\partial \alpha} f(i)
\]

(38)

\[
0 \approx \delta \frac{\partial y(i)}{\partial \alpha} + \delta \frac{\partial^2 y(i)}{\partial \alpha \partial \beta} + \delta b \sum \frac{1}{u(i)} \frac{\partial y(i)}{\partial \alpha} \frac{\partial y(i)}{\partial \beta}
\]

(39)

Equating Eqs. 38 and 39 along with analogous equations for the partial derivative with respect to \(b\) then give a set of simultaneous equations to
solve for the desired unknowns \((\delta a, \delta b)\) as functions of \((\epsilon, y(i), u(i), f(i))\). For the general case of \(m\) parameters \((a_1, \ldots, a_m)\)

\[
\sum_m M_{nm} \delta a_m = \epsilon B_n
\]  
(40)

\[
M_{nm} = \sum \frac{1}{u(i)} \frac{\partial y(i)}{\partial a_n} \frac{\partial y(i)}{\partial a_m}
\]  
(41)

\[
B_n = \sum \frac{1}{u(i)} \frac{\partial y(i)}{\partial a_n} f(i)
\]  
(42)

5.2 Systematic Fitting Errors in the Ratio Formulation

Using Eqs. 40 - 42 applied to the case of the ratio fitting procedure and assuming a two-parameter parent distribution \(D(\omega_0, \phi)\) give an approximate formula for the systematic error in fitting for \(\omega_0\) as

\[
\frac{\delta \omega_0}{\omega_0} \approx \left( \frac{\epsilon}{2\pi \Lambda_\alpha} \right) \left( \frac{\tau_\alpha}{\tau_\mu} \right) \left( \frac{cd - be}{ac - bb} \right)
\]  
(43)

where

\[
a = \sum \frac{x^2 e^{-x} \sin^2(\kappa x + \phi)}{1 - A_0^2 \cos^2(\kappa x + \phi)}
\]  
(44)

\[
b = \sum \frac{x e^{-x} \sin^2(\kappa x + \phi)}{1 - A_0^2 \cos^2(\kappa x + \phi)}
\]  
(45)

\[
c = \sum \frac{e^{-x} \sin^2(\kappa x + \phi)}{1 - A_0^2 \cos^2(\kappa x + \phi)}
\]  
(46)

\[
d = \sum \frac{x e^{-x} \sin(\kappa x + \phi)}{1 - A_0^2 \cos^2(\kappa x + \phi)} f(x)
\]  
(47)

\[
e = \sum \frac{e^{-x} \sin(\kappa x + \phi)}{1 - A_0^2 \cos^2(\kappa x + \phi)} f(x)
\]  
(48)

with \(x\) and \(k\) defined by

\[
x = \frac{t}{\tau_\mu}
\]  
(49)

\[
k = 2\pi \left( \frac{\tau_\mu}{\tau_\alpha} \right)
\]  
(50)
The sums in Eqs. 44 – 48 are to be taken over the time range of the fit. Note that the sums \((a, b, c)\) are of \(O(\frac{1}{\wedge})\) and are slowly changing as a function of fit start time, where as the time behavior of \((d, e)\) depend on the functional form of the background term \(f(x)\).

Consider first the case of neglecting the additive constant term \(\epsilon\) in the ratio functional (see Eq. 13). In this case, \(f(x) = 1\), and the sums \((d, e)\) are sinusoidal versus fit start times. From Eq. 43, \((\delta\omega_n / \omega_n)\) versus fit start time is then also sinusoidal with frequency \(\omega_n\). Figure 10 shows the error \((\delta\omega_n / \omega_n)\) as a function of fit start times in the range 25 – 45 \(\mu s\) using three different methods. The first method is analytical using Eq. 43. In the second method one set of simulated data at \(\approx 1.2\) ppm statistical uncertainty was fit using the ratio procedure with and without the \(\epsilon\) term, and the differences between these two fits were used to estimate the error in neglecting the \(\epsilon\) term. In the last method data was simulated without any statistical fluctuation (i.e. \(\delta\mu_i = 0\) in the algorithm outlined in section 4.1), and the difference between the fit and input frequency was used to estimate the error in neglecting \(\epsilon\). The agreement among the three methods is very good. Hence, any one of these three methods may be used to estimate fitting systematic errors. As a final check, fits to 1600 independent sets of simulated data at start times 25 and 26 \(\mu s\) were averaged and the values compared to the case of fitting data simulated without statistical fluctuation. The results are shown in figure 11. It should be added that at later fit start times the fitting errors are as shown in figures 10 and 11 with the amplitude of the sinusoidal oscillation slowly decreasing in time.

Consider next the case of neglecting the term \(-\epsilon\lambda(t)\) in the ratio functional (see Eq. 10). In this case, \(f(x) = -\lambda^2 \cos^2(kx + \phi)\), and the error \((\delta\omega_n / \omega_n)\) is expected to behave like \(\cos^3\) versus fit start time (since the sum \(\sum \sin * \cos^2 \rightarrow \cos^3\)). Figure 13 shows a plot of the error \((\delta\omega_n / \omega_n)\) as a function of fit start time using both analytical calculations and fits to data simulated without statistical fluctuations. Note that the systematic error is \((\delta\omega_n / \omega_n) < 0.01\) ppm, as mentioned previously in section 2.

6 Summary

In this note, the ratio fitting formulation was described. In sections 2 and 3, the formulation of the ratio fitting procedure was detailed. It was shown that fitting to the time spectrum described by the five-parameter function
\[ n(t) = N_0 e^{\frac{t}{\tau_0}} (1 - A_0 \cos(\omega_0 t + \phi)) \]  \hspace{1cm} (51)

can be reformulated to the ratio fitting procedure of minimizing the \( \chi^2 \) sum

\[ \chi^2 = \sum_{i=1}^{N} \left( u_i \log_2 R_i + v_i \log_2 (1 - R_i) \right)^2 \]  \hspace{1cm} (52)

where the ratio functional \( R(t) \) is

\[ R(t) = A_0 \cos(\omega_0 t + \phi) + \frac{1}{16} \left( \frac{\tau_0}{\tau_\mu} \right)^2 \]  \hspace{1cm} (53)

In sections 4 and 5, the quantitative characteristics of the ratio fitting procedure was studied in detail using both fits to simulated data and analytical techniques. At the statistical level of the 1999 data analysis (~ 1.2 ppm uncertainty in \( \omega_0 \)), the distribution of \( \chi^2 / \text{ndf} \) for the ratio fitting procedure is centered at \( \approx 10007 \) with width \( \approx 0.96(\sqrt{2/\text{ndf}}) \), and the systematic uncertainty in the fitted value of \( \omega_0 \) is 0.05 ppm. The systematic error in the fitted value of \( \phi \) is \( \approx 0.05 \) mrad. Due to the half-period shifting \( \pm (\tau_0 / 2) \) in the ratio formulation, there is a statistical fluctuation between the fitted values of \( \omega_0 \) for the normal five-parameter fits and the ratio fits. The width of that distribution was found to be \( \approx 13% \) of the statistical uncertainty.

It should be noted that the quoted systematic uncertainty for the ratio fitting formulation is only for the procedure itself. Additional uncertainties due to background terms such as pileup, coherent betatron oscillations, etc., will be documented in a later g-2 note.

References

Figure 1: (a) Distribution of $R = (\omega_{FIT} - \omega_{MC})/(\omega_{MC})$ for ratio fits to 1600 distinct sets of simulated data; the fitting range is 25 – 600 $\mu$s, (b) Distribution of the statistical uncertainty in $R$, $\sigma_R = \sigma_{\omega_{FIT}}/\omega_{MC}$, for the 1600 fits in (a).
Figure 2: Distribution of $\chi^2/\text{ndf}$ for ratio fits to 1600 distinct sets of simulated data; the fitting range is $25 - 600 \mu s$.

Figure 3: Mean value of $\mathcal{R}$ versus fit start time; the one sigma correlated error bands are plotted for the fit start time of $25 \mu s$. 

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Figure 4: Mean value of $\chi^2/ndf$ versus fit start time.

Figure 5: Distribution of fitted asymmetries for ratio fits to 1600 independent sets of simulated data with input asymmetry 0.36; the fitting range is $25 - 600 \mu s$. 
Figure 6: Distribution of fitted phases $\phi$ for ratio fits to 1600 independent sets of simulated data with input phase 0; the fitting range is $25 - 600$ $\mu$s.

Figure 7: Distribution of $R$(five-parameter fit) – $R$(ratio fit) for 1600 independent sets of simulated data; the fitting range is $25 - 600$ $\mu$s.
Figure 8: (a) Distribution of $R$ for five-parameter fits to 1600 distinct sets of simulated data with binwidth 136.4375 $ns$; the fitting range is $25 - 600 \mu s$, (b) Distribution of the statistical uncertainty in $R$ for the 1600 fits in (a).
Figure 9: (a) Distribution of $R$ for ratio fits to 1600 distinct sets of simulated data with binwidth 136.4375 ns; the fitting range is 25 – 600 $\mu$s, (b) Distribution of the statistical uncertainty in $R$ for the 1600 fits in (a).
Figure 10: $R = (\omega_{\text{FIT}} - \omega_{\text{MC}})/\omega_{\text{MC}}$ versus fit start time for the ratio functional $R(t) = A_0 \cos(\omega_0 t + \phi)$; the terms $(\epsilon - \epsilon(t))$ are neglected.

Figure 11: $R = (\omega_{\text{FIT}} - \omega_{\text{MC}})/\omega_{\text{MC}}$ versus fit start time for the ratio functional $R(t) = A_0 \cos(\omega_0 t + \phi)$; the terms $(\epsilon - \epsilon(t))$ are neglected.
\[ R = \frac{(\omega_{FIT} - \omega_{MC})}{\omega_{MC}} \]

versus fit start time for the ratio functional \( R(t) = A_0 \cos(\omega t + \phi) + \epsilon \); the term \(-\epsilon \lambda(t)\) is neglected.

Figure 12: \( R = (\omega_{FIT} - \omega_{MC})/\omega_{MC} \) versus fit start time for the ratio functional \( R(t) = A_0 \cos(\omega t + \phi) + \epsilon \); the term \(-\epsilon \lambda(t)\) is neglected.