Lattice calculation of the lowest-order hadronic contribution to the muon anomalous magnetic moment

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[T. Blum, hep-lat/0212018]

OUTLINE of the talk

1. Introduction to muon g-2 and hadronic contributions
2. Theory of the Euclidean space-time calculation
3. Lattice calculation of the vacuum polarization
4. Results for the lowest order (α^2) hadronic contribution
5. Summary and Outlook
6. The (α^3) hadronic light by light calculation

I. Introduction to muon g-2...

Classical interaction of particle with static magnetic field

$$V(\vec{x}) = -\mu \cdot \vec{B}$$

The magnetic moment μ is proportional to its spin

$$\vec{\mu} = g \left(\frac{e}{2m} \right) \vec{S}$$

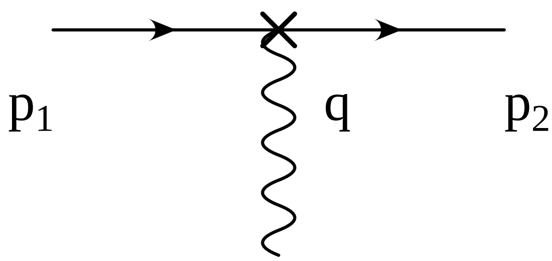
The Landé g-factor is predicted from the Dirac eq. to be

$$g = 2$$

for elementary fermions

In the quantum theory this will change.

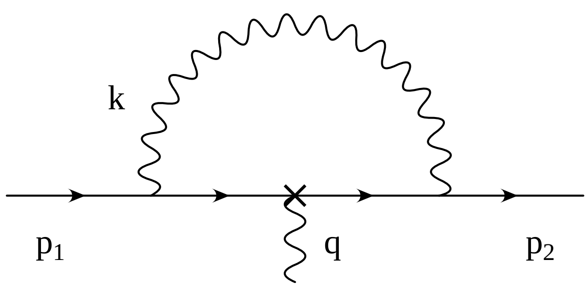
At tree-level in perturbation theory we have



$$\rightarrow\ -i\, e\, \bar{u}(p_2)\, \gamma^\mu\, u(p_1)\ \cdot A^{\mathrm{cl}}_\mu(p_2 - p_1),$$

where $q = p_2 - p_1$, which gives $g = 2$

but radiative corrections modify the vertex



$$\gamma_\mu\ \rightarrow\ \Gamma^\mu(q)\ =\left(\gamma^\mu\, F_1(q^2) + \frac{i\,\sigma^{\mu\nu}\, q_\nu}{2m}\, F_2(q^2)\right)$$

which results from Lorentz invariance and current-conservation (Ward-Takahashi identity) when the muon is on-shell.

Form factors F_1 and F_2 contain all information about muon's interaction with the electromagnetic field. In particular, it's charge and magnetic moment.

$F_1(0) = 1$ is charge of the muon in units of e. Since $F_1^{\text{tree}}(0) = 1$, all radiative corrections vanish (W-T identity).

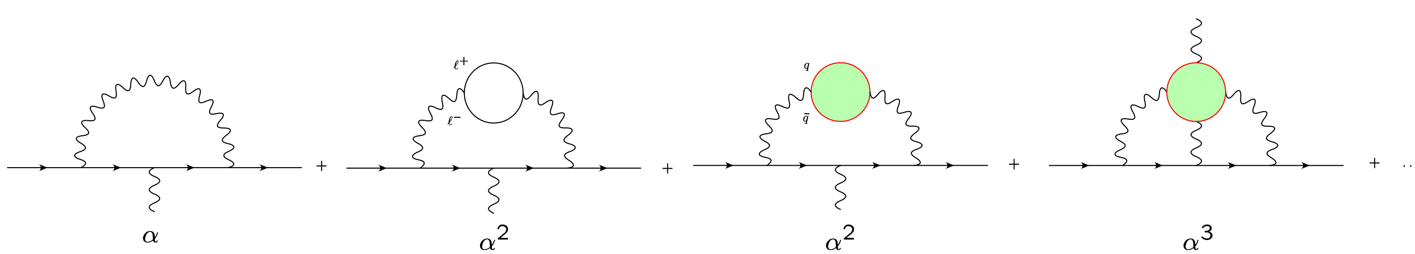
There is no F_2 term at tree level. In general,

$$g = 2(F_1(0) + F_2(0))$$
$$F_2(0) = \frac{g-2}{2} \equiv a_\mu$$

and corrections start at $\mathcal{O}(\alpha)$.

Compute these corrections order-by-order in perturbation theory by expanding $\Gamma^\mu(q^2)$ in QED coupling constant

$$\alpha = \frac{e^2}{4\pi} = \frac{1}{137} + \ldots$$



Theory status

- QED. $a_\mu^{QED} = \sum C_n(\alpha/\pi)^n \quad (n = 1\text{-}5)$

$C_1 = 1/2$ (Schwinger term (1948) $\alpha/(2\pi) = .00116\ 14...$)

$$a_\mu^{\mathrm{QED}}(\mathrm{total}) = 11\,658\,470.57(29) \times 10^{-10}$$

Kinoshita, *et al.*, ...

- Hadronic. $a_\mu^{\mathrm{had}}(\alpha^2)(e^+e^-\ \mathrm{data}) = 684.7(6.0)(3.6) \times 10^{-10}$

Davier, *et al.* (2002); Hagiwara *et al.* (2002)

$$a_\mu^{\mathrm{had}}(\alpha^2)(\tau\ \mathrm{decay\ data}) = 709.0(5.1)(1.2)(2.8) \times 10^{-10}$$

Davier, *et al.*(2002)

$$a_\mu^{\mathrm{had}}(\alpha^3) = -10.0(6) \times 10^{-10}$$

$$a_\mu^{\mathrm{had}}(\alpha^3\ \mathrm{light-by-light}) = 8.6(3.2) \times 10^{-10}$$

Knecht, *et al.*(2002), ...

- Electroweak. $a_\mu^{\mathrm{EW}}(\mathrm{total}) = 15.4(1)(2) \times 10^{-10}$

Czarnecki, *et al.*(2002), Knecht, *et al.*(2002)

- Theory total: $a_\mu^{\mathrm{Theory}(e^+e^-)} = 11\,659\,169.1(7.0)(3.5)(0.3) \times 10^{-10}$

$$a_\mu^{\mathrm{Theory}(\tau\ \mathrm{decay})} = 11\,659\,193.6(5.9)(3.5)(0.3) \times 10^{-10}$$

Davier, *et al.*(2002)

The recent precise measurement at Brookhaven (E821) allows a precision test of the Standard Model

- Experimental value: $a_\mu^{\text{Exp}} = 11\,659\,204(7)(5) \times 10^{-10}$

  Muon (g-2) Collab., BNL (2002)

  accuracy to better than 1 ppm!

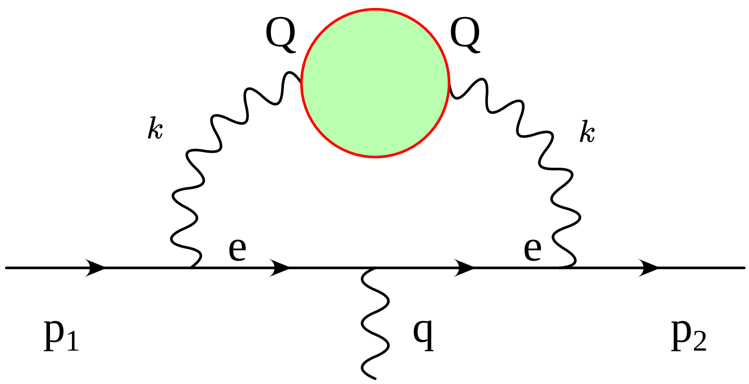
- Hadronic contribution dominates theory error
- e^+e^-: 3 σ deviation with experiment
- τ decay: 0.9 σ deviation with experiment

Taking the difference of the two theory calculations, the uncertainty in hadronic contribution could be 5%

Motivation to attempt first principles lattice calculation of the hadronic contribution (really, first principles nature is enough)

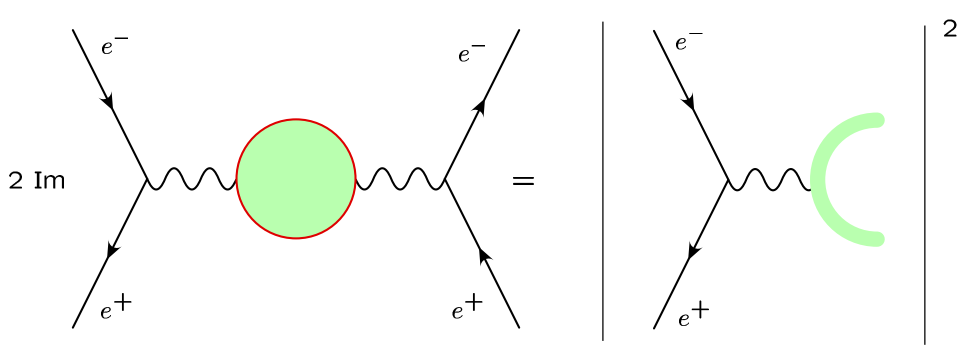
Exactly the kind of calculation the lattice is supposed to handle

Contribution we are after: $\mathcal{O}(\alpha^2)$ hadronic vacuum polarization: $\Pi(k^2)$



But the blob, which represents all possible intermediate hadronic states, is not calculable in perturbation theory (QCD!)

From the optical theorem and the analytic structure of $\Pi(k^2)$, we can compute it from experimental data for $e^+e^- \rightarrow$ hadrons cross-section (and τ decay branching ratios).



The usual dispersive method

[Bouchiat and Michel (1961); Durand (1962); ...]

The vacuum polarization is an *analytic* function.

$$\Pi(q^2) = \frac{1}{\pi}\int_0^\infty \mathrm{d}s \frac{\mathrm{Im}\Pi(s)}{(s-q^2)}$$

$$\sigma_{\mathrm{total}}(e^+e^- \to \mathrm{hadrons}) = \frac{4\pi^2\alpha}{s}\frac{1}{\pi}\mathrm{Im}\Pi(s)$$

(by the optical theorem) which leads to

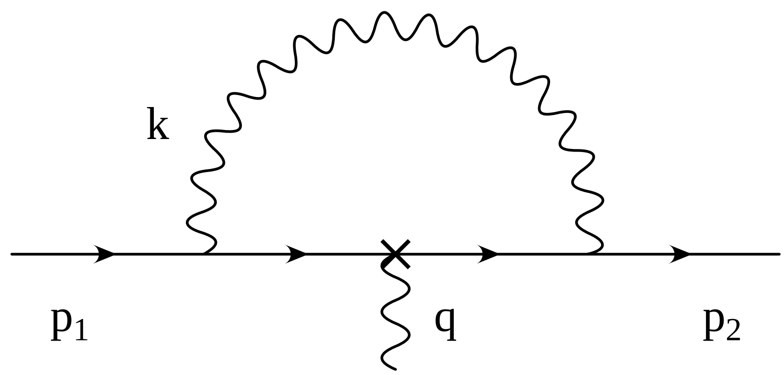
$$a_\mu^{\mathrm{had}(2)} = \frac{1}{4\pi^2}\int_{4m_\pi^2}^\infty \mathrm{d}s K(s)\sigma_{\mathrm{total}}(s)$$

where where $K(s)$ is a known function

$K(s)$ is strongly weighted to low energy region: roughly 91% from $\sqrt{s} \lesssim 1.8$ GeV, 73% from two pion final state which is dominated by the $\rho(770)$ resonance.

2. Theory of the Euclidean space-time calculation

Start in Minkowski space-time, omit the quark loop for the moment, lowest order in QED contribution



$$\Gamma^{(1)}_\mu = e^2\, i \int \frac{d^4 k}{(2\pi)^4}\, \gamma_\nu\, \frac{-i\,\not{p}_2 + i\,\not{k} + m_\mu}{(p_2 - k)^2 + m_\mu^2 - i\,\epsilon}\, \gamma_\mu \times \frac{-i\,\not{p}_1 + i\,\not{k} + m_\mu}{(p_1 - k)^2 + m_\mu^2 - i\,\epsilon)}\, \gamma_\nu\, \frac{1}{k^2 - i\,\epsilon}$$

where $p_{1,2} = p \mp q/2$

Project $F_2(q^2)$ from Γ_μ:

$$P_\rho \;=\; (-i\,\not{p}_1 + m_\mu)\left(g_1\gamma_\rho + \frac{i}{2\,m_\mu} g_2\,(p_1+p_2)_\rho\right)(-i\,\not{p}_2 + m_\mu)$$

$$F_2(q^2) \;=\; \mathrm{Tr}\left[(-i\,\not{p}_1 + m_\mu)\left(-\gamma_\rho + i\frac{-q^2+2\,m_\mu^2}{m_\mu(-q^2-4m_\mu^2)}(p_1+p_2)_\rho\right)\right.$$
$$\left.\times\;(-i\,\not{p}_2 + m_\mu)\,\Gamma_\rho\,\right]\frac{m_\mu^2}{-q^2(-q^2+4m_\mu^2)}$$

taking the limit $q^2 \to 0$, we find ($a_\mu^{(1)} = F_2(0)$)

$$a_\mu^{(1)} \;=\; e^2\,i\int\frac{d^4\,k}{(2\pi)^4}\frac{1}{((p-k)^2+m_\mu^2-i\,\epsilon)^2}\frac{1}{k^2-i\,\epsilon}$$
$$\times\;\left(\frac{16\,(p\cdot k)^2}{3\,m_\mu^2}+\frac{4}{3}\,k^2+4\,(p\cdot k)\right)$$

To do the integral, analytically continue to Euclidean space-time

$$k_0 \rightarrow iK_4$$
$$p_0 \rightarrow iP_4 \quad (P^2 \text{ now space} - \text{like})$$

and transform to 4d polar coordinates

$$\int \mathrm{d}^4K \rightarrow \int \mathrm{d}\Omega_K \int_0^\infty \frac{1}{2} K^2 \, \mathrm{d}K^2$$

Use Hypersherical Polynomials $C_n(\hat{P} \cdot \hat{K})$ (see, *e.g.*, Handbook of Mathematical Functions, Stegun and Abramowitz) to expand quark denominator

[Roskies, R. Z. and Levine, M. J. and Remiddi, E., (1990)]

$$\left(\frac{1}{(K-P)^2 + m_\mu^2}\right)^2 = \frac{Z^2}{1 - P^2K^2Z^2} \sum_{n=0}^{\infty} (n+1)(PKZ)^n \, C_n(\hat{P} \cdot \hat{K})$$

and numerator,

$$\hat{P} \cdot \hat{K} = \frac{1}{2} C_1(\hat{P} \cdot \hat{K}) \ , \ (\hat{P} \cdot \hat{K})^2 = \frac{1}{4} C_0(\hat{P} \cdot \hat{K}) + C_2(\hat{P} \cdot \hat{K})$$

where

$$Z = \frac{K^2 + P^2 + m_\mu^2 - [(K^2 + P^2 + m_\mu^2)^2 - 4P^2K^2]^{1/2}}{2 \, P^2 \, K^2}$$

Using the orthogonality relation

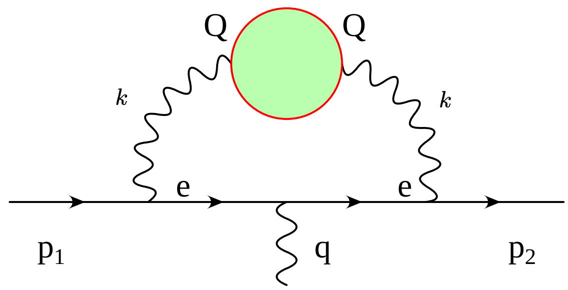
$$\int \frac{d\Omega_{\hat{K}}}{2\pi^2} C_m(\hat{P}\cdot\hat{K}) C_n(\hat{P}\cdot\hat{K}) \;=\; \frac{\delta_{m,n}}{n+1} C_n(\hat{P}\cdot\hat{P})$$

we can easily do the the angular integrations and the sum over n. Continuing $P^2 \to -m_\mu^2$ back on-shell, we are left with

$$\begin{aligned} a_\mu^{(1)} \;&=\; \frac{\alpha}{\pi}\int_0^\infty \mathrm{d}K^2 \frac{m_\mu^2 K^2 Z^3 (1 - K^2\, Z)}{1 + m_\mu^2 K^2\, Z^2} \\ &=\; \frac{\alpha}{\pi}\int_0^\infty \mathrm{d}K^2\, f(K^2), \end{aligned}$$

Easy to verify Schwinger contribution $a_\mu^{(1)} = \alpha/2\,\pi$.

Now, back to the $\mathcal{O}(\alpha^2)$ hadronic vacuum polarization.



Adding the quark loop (blob) is trivial by noting that only the photon propagator is modified (*c.f.*, charge renormalization in QED),

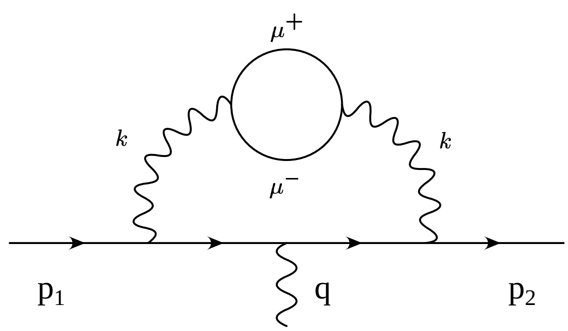
$$\frac{1}{k^2 - i\,\epsilon} \to \frac{1}{k^2 - i\,\epsilon} \times (1 + \Pi(k^2) + \ldots)$$

Crucial that we were able to set up the one-loop integral so that the photon propagator only depended on the loop momentum K^2. So, we can use the lattice calculation of $\Pi(K^2)$ to obtain the $\mathcal{O}(\alpha^2)$ hadronic contribution.

$$a_\mu^{(2)\mathrm{had}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dK^2\, f(K^2)\, \Pi(K^2)$$

A simple check

Insert the one-loop QED vacuum polarization (see, *e.g* Peskin and Schroeder),



analytically continued to Euclidean space, into the expression for $a_\mu^{(2)\mathrm{had}}$ and compare the result to the known value of the $\mathcal{O}(\alpha^2)$ QED contribution to a_μ. It agrees.

Finally, note that the kernel $f(K^2)$ diverges as $K^2 \to 0$, so the integral is dominated by the low momentum region.

3. Lattice calculation of the vacuum polarization

The vacuum polarization is defined as

$$\Pi^{\mu\nu}(q) = \int \mathrm{d}^4x\, \mathrm{e}^{i\,q\,(x-y)}\, \langle J^\mu(x) J^\nu(y)\rangle \qquad (J^\mu(x) = \bar\psi\gamma_\mu\psi(x))$$
$$= (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$$

and satisfies the Ward-Takahashi identity (charge conservation)

$$q_\mu\, \Pi^{\mu\nu}(q) = 0 \qquad (\partial_\mu\, J^\mu = 0)$$

In the lattice regularization using domain wall fermions (DWF), current conservation is given by

$$\Delta^\mu J^\mu(x) = \sum_\mu \frac{J^\mu(x) - J^\mu(x - a\hat\mu)}{a}$$

where

$$J^\mu(x) = \sum_s \frac{1}{2}\Big(\bar\psi(x+\hat\mu, s)U^\dagger(x)(1+\gamma^\mu)\psi(x,s)$$
$$- \bar\psi(x,s)U(x)(1-\gamma^\mu)\psi(x+\hat\mu,s)\Big)$$

For the two-point function this yields

$$\Delta^\mu J^\mu(x)\,(J^\nu(y))^\dagger =$$

$$- \sum_s \delta(x-y)\frac{1}{2}\left(\bar\psi(y+\hat\nu,s)U^\dagger(y)(1-\gamma^\nu)\psi(y,s)+\bar\psi(y,s)U(y)(1+\gamma^\nu)\psi(y+\hat\nu,s)\right)$$
$$+ \; \delta(x-y-\hat\nu)\frac{1}{2}\left(\bar\psi(y+\hat\nu,s)U^\dagger(y)(1-\gamma^\nu)\psi(y,s)+\bar\psi(y,s)U(y)(1+\gamma^\nu)\psi(y+\hat\nu,s)\right),$$

(valid on any gauge-field configuration). The contact terms occur because the lattice conserved current is *point-split*. After subtracting [Gockeler, *et al.* (2001)]

$$\delta^{\mu\nu}\sum_s \frac{1}{2}\left(\,\bar\psi(y+\hat\nu,s)U^\dagger(y)(1-\gamma^\nu)\psi(y,s)+\,\bar\psi(y,s)U(y)(1+\gamma^\nu)\psi(y+\hat\nu,s)\,\right)$$

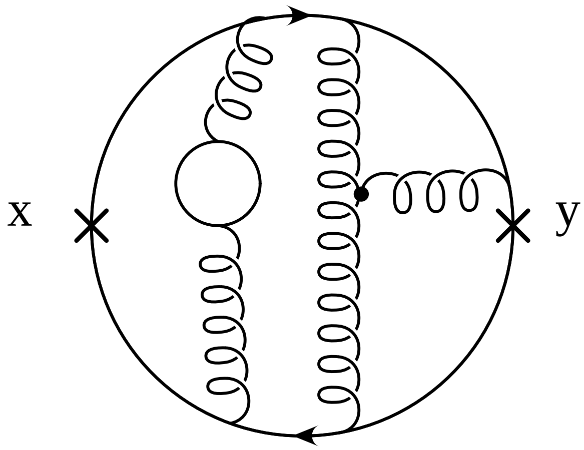
to cancel the contact terms, Fourier transformation of the two point function yields the usual W-T identity

$$\begin{aligned}\hat q^\mu\Pi^{\mu\nu}(\hat q^2) &= 0\\ \hat q^\mu &= \frac{2}{a}\sin\left(\frac{aq^\mu}{2}\right)\\ q^\mu &= \frac{2\pi \mathsf{n}_\mu}{a\,L_\mu},\quad \mathsf{n}_\mu = 0,\ldots,L_\mu-1\end{aligned}$$

where

$$\Pi^{\mu\nu}(\hat q) = (\hat q^\mu\hat q^\nu - \hat q^2\delta^{\mu\nu})\Pi(\hat q^2)$$

In practice, calculate the two point function in coordinate space on a finite Euclidean space-time lattice



Wick contract quark fields into propagators ($M^{-1}_{x,y}$),

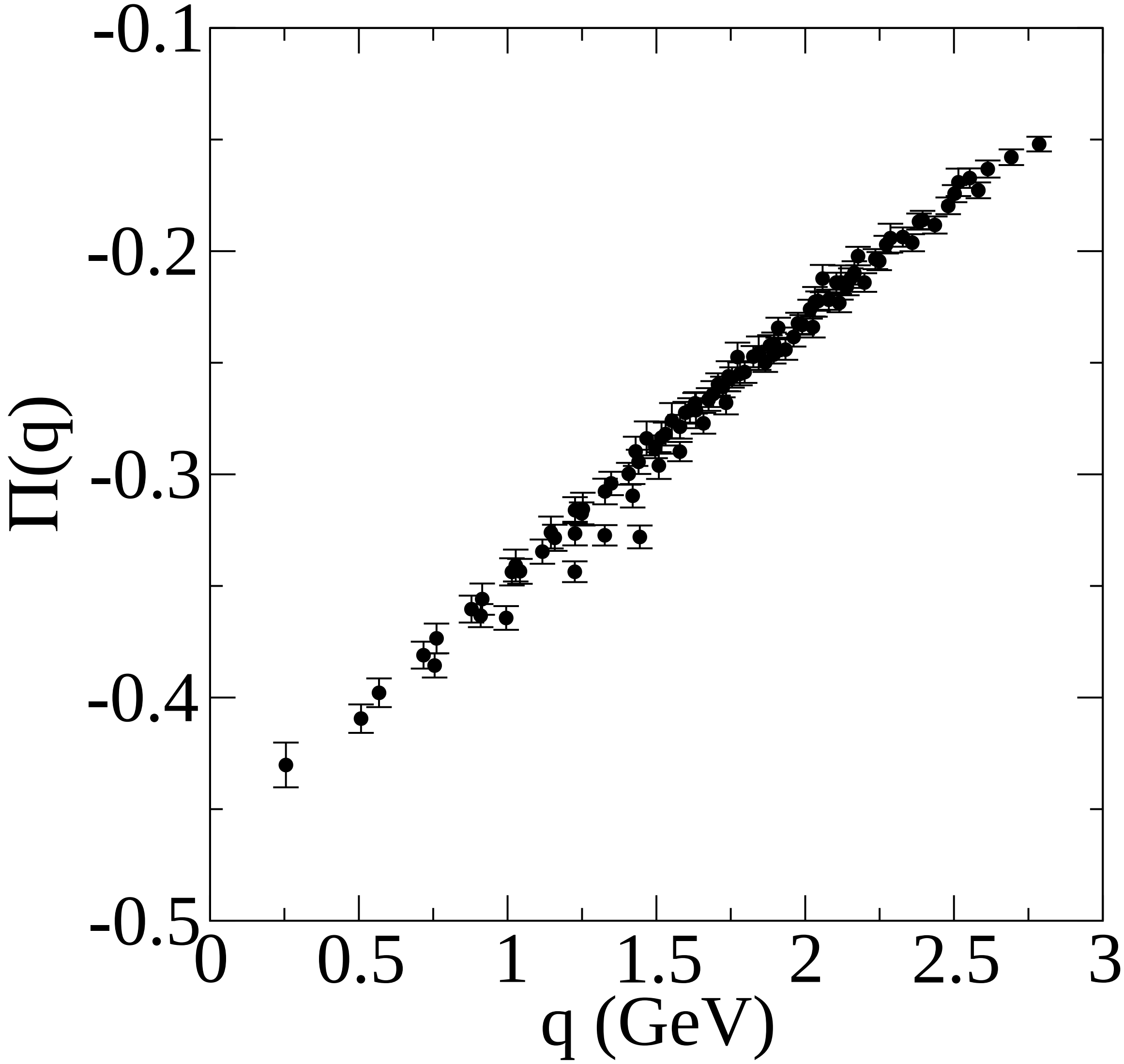
$$C^{\mu\nu}(x,y) \;=\; \mathrm{Tr} M^{-1}_{x,y}\,\gamma^{\mu}\,M^{-1}_{y,x}\,\gamma^{\nu}$$

Then compute the Fourier transform

$$\Pi^{\mu\nu}(\hat{q}) \;=\; \sum_{x} e^{i\,q(x-y)}\,C^{\mu\nu}(x,y),$$

and average over many gauge-field configurations

note: no virutal quark loops in the quenched approx.

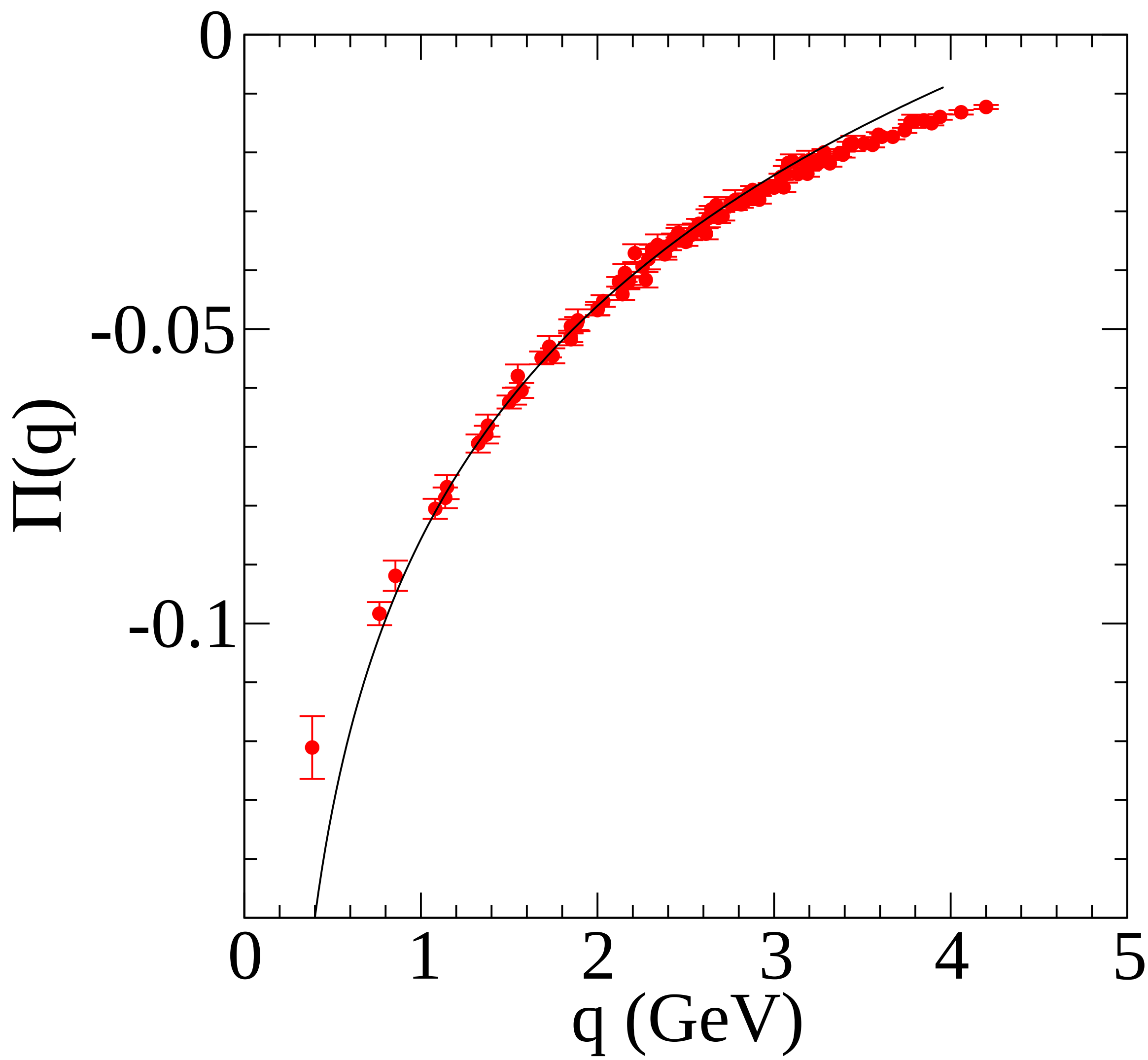


Using point-split conserved current, so unphysical heavy $5d$ modes also contribute

Large $\mathcal{O}(aq)$, $\mathcal{O}(aq)$ errors are clearly visible

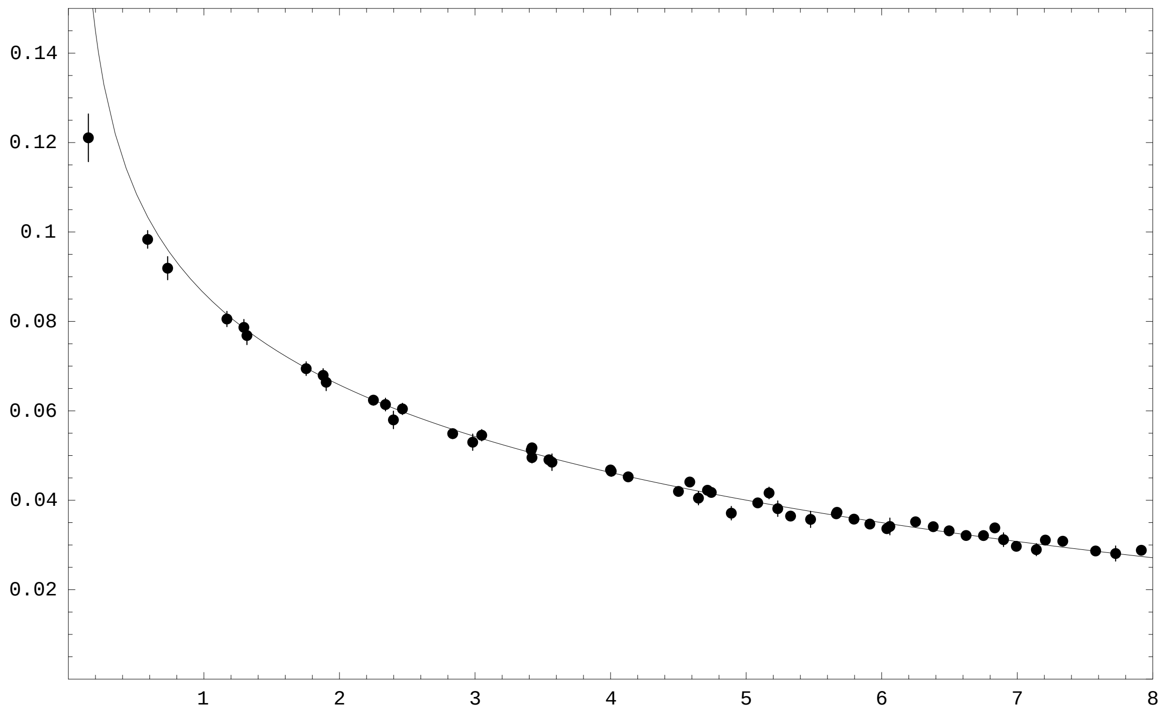
These effects are easily subtracted

After subtraction $\mathcal{O}(aq)$ eliminated and $\mathcal{O}(a^2)$ errors reduced.



Points: DWF+DBW2 gauge action, $1/a \approx 1.3$ GeV, $m_f = 0.04$ (≈ 90 MeV), volume∼ $(2.4$ fm$)^3$

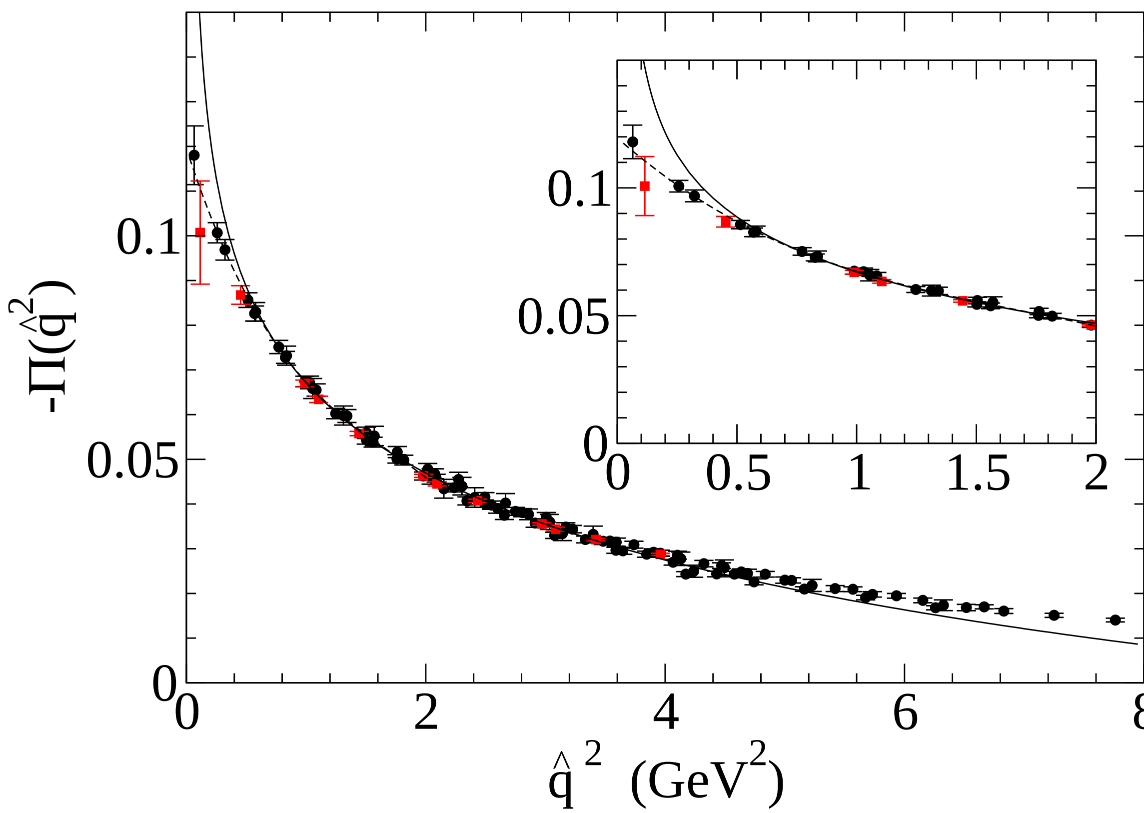
Solid curve: 3-loop perturbation theory ($\overline{MS}$, $\mu = 1/a = 1.3$ GeV, $m = Z_m m_f$, $Z_m^{\overline{MS}}(1.3\,\text{GeV}) \approx 1.7$) [Chetyrkin, *et al.* (1996)]



Points: DWF+DBW2 gauge action, $1/a \approx 2$ GeV, $m_f = 0.04$, volume$\sim (1.6 \text{ fm})^3$

Solid curve: 3-loop perturbation theory ($\overline{MS}$, $\mu = 1/a = 2$ GeV, $m = Z_m m_f$, $Z_m^{\overline{MS}}(2\,\text{GeV}) \approx 1.4$) [Chetyrkin, *et al.* (1996)]

Volume comparison on $a^{-1} \approx 1.3$ GeV lattice



Black circles: volume$\sim (2.4$ fm$)^3$

Red Squares: volume$\sim (1.2$ fm$)^3$

Finite volume corrections significant at low $\hat{q}^2$

4. Results for the lowest order hadronic contribution

In general $\Pi(q^2)$ is logarithmically divergent ($e.g.$, $\sim \ln a$). Conventional renormalization is to choose $\Pi(0) = 0$, *i.e.*

$$\hat{\Pi}(q^2) \equiv \Pi(q^2) - \Pi(0)$$

$\Pi(q^2)$ is regular as $q^2 \to 0$, fit to ansatz:

$$\Pi(q^2) = a_0 + a_1\,q^2 + a_2\,q^4 + a_3\,q^6 + \ldots$$

(improved by, *e.g.* chiral pert. theory? Practical question)

This reasonably reproduces the lattice data

plug into $\int_0^\infty \mathrm{d}K^2\, f(K^2)\hat{\Pi}(K^2)$

Numerically integrate (in MATHEMATICA) from $0 \to K^2_{\text{cut}}$

Use perturbation theory for $K^2_{\text{cut}} \to \infty$

Contribution of (degenerate) up, down, and strange quarks

$$\left(\sum_i Q_i^2\right)\hat{\Pi}(q^2) \;=\; \frac{2}{3}\,\hat{\Pi}(K^2)$$

a^{-1}	V	$a_\mu^{\text{had}}(\alpha^2)$	
1.3	(2.4 fm)^3	460(78) x10^{-10}	
1.3	(1.2 fm)^3	318(69) x10^{-10}	
2.0	(1.6 fm)^3	378(96) x10^{-10}	
∞	∞	684.7(6)(3.6) x10^{-10}	e^+e^-
∞	∞	709.0(5.1)(1.2)(2.8) x10^{-10}	τ decay

Recall 72% of dispersive result came from the $\rho(770)$ resonance which is all there is in quenched case (zero width).

Errors on lattice results are statistical only (# conf $\lesssim$ 25).

low q^2 region dominates; work harder on statistics

Large finite volume effect

quark mass unphysically large

quenching error

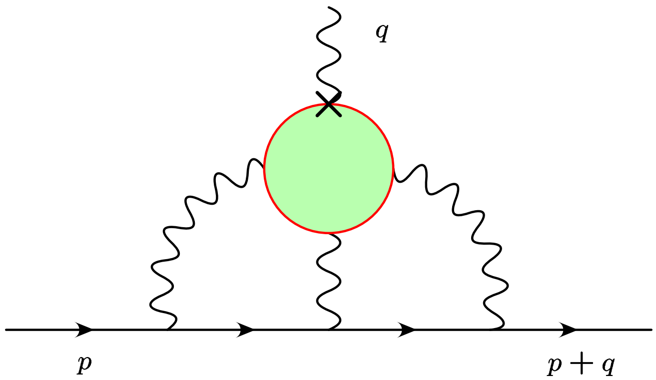
Pert. contribution is roughly 1-10 x10^{-10}

5. Summary and Outlook

- muon g-2 is an important precision test of the Standard Model (*c.f.*, recent measurement at BNL)
- Hadronic contribtutions biggest theoretical uncertainty
- First princliples (lattice) calculation not only possible, maybe useful (Comp.: $\lesssim$ 2 weeks on 1/6 QCDSP at BNL)
- Calculations with improved K-S fermions in progress on MILC lattices (not too hard)
  - 2+1 flavors of quarks (unquenched)
  - more physical quark masses: $m_u = m_d \approx (1/10 - 1/5) \times m_s$
  - Larger volume (2.5-3.0 fm boxes, $a^{-1} \approx$ 1.6-2.2 GeV)
  - Improve statistics (# avail. configs $\gtrsim$ 400 @ each point)
  - discretization errors?
- Future: Dynamical DWF, hadronic light-by-light (α^3) contribution (more challenging, but required precision not as high)

The hadronic light-by-light contribution (α^3)

Calculating as before likely difficult. Instead, compute completely non-perturbatively, including photons. *i.e.* average over $SU(3) \times U(1)$ gauge fields



But, need to subtract α^2 contribution