NO BOOKS or NOTES ARE PERMITTED

\[ A_{as}(\omega) = \frac{F_{3/m}}{[\omega_0^2 - \omega^2 + \gamma^2 \omega^2]^\frac{3}{2}} \quad \text{tan} \phi_D = \frac{-\gamma \omega}{\omega_0^2 - \omega^2} \]

\[ \frac{1}{2} \frac{L_i^2}{c}, \quad \frac{1}{2} \frac{L_i^2}{c} \quad 1 + \tan^2 x = \sec^2 x \quad \text{where} \quad 1/ \cos x = \sec x \]

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \quad \cos 2x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x
\]

Binomial: \( (1 + x)^n \approx 1 + nx + \frac{n(n-1)x^2}{2!} + \sum_{j=4}^{\infty} c_j x^{j-1} \quad |x| < 1; \quad c_j = \frac{n(n-1)\cdots(n-(j-2))}{(j-1)!} \]

Taylor: \( f(x) = f(a) + (x - a)f'(a) + \frac{1}{2!}(x - a)^2f''(a) + \ldots + \frac{1}{n!}(x - a)^nf^n(a) \)

Fourier: \( \psi(t) = A_0 + \sum_{n=1}^{\infty} a_n \cos(n \omega t) + \sum_{n=1}^{\infty} b_n \sin(n \omega t) \quad A_0 = \frac{1}{\tau} \int_{-\tau}^{\tau} \psi(t) \, dt \quad \tau \Rightarrow 1 \text{ period} \)

\[
a_m = \frac{2}{\tau} \int_{-\tau}^{\tau} \psi(t) \cos(n \omega t) \, dt; \quad b_n = \frac{2}{\tau} \int_{-\tau}^{\tau} \psi(t) \sin(n \omega t) \, dt; \quad \psi(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{i n \omega t} \quad c_0 = A_0
\]

\[
c_n = \frac{1}{\tau} \int_{-\tau}^{\tau} \psi(t) e^{-i n \omega t} \, dt; \quad \psi(t) = \int_{-\infty}^{\infty} C(\omega) e^{i \omega t} \, d\omega; \quad C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(t) e^{-i \omega t} \, dt
\]

\[
\int x \cos ax \, dx = \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax; \quad \int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax
\]

\[
\int \ln (ax + b) \, dx = \frac{ax + b}{a} \ln (ax + b) - x
\]

\[
\int \frac{dv}{(1 - v^2)^\frac{3}{2}} = c \tanh^{-1} \frac{v}{c} \quad \text{where} \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}
\]
For the simplest nonlinear oscillator, there are two possible cases:

i). The force is symmetric about the equilibrium point. If the force is represented by

\[ F = - (1 + \alpha x^2) \, sx \]

then the simplest form of the solution is

\[ x = A (\cos \omega_1 t + \epsilon \cos 3\omega_1 t) \]

we find in lowest order

\[ \omega_1 \simeq \omega_0 \left( 1 + \frac{3}{8} \alpha A^2 \right) \]

and

\[ \epsilon \simeq \frac{1}{32} \alpha A^2 \]

ii). The force is not symmetric about the equilibrium point and the simplest form for the force is

\[ F = - (1 + \beta x) \, sx \]

and the simplest form of the solution is

\[ x = A_0 + A \, (\cos \omega_1 t + \delta \cos 2\omega_1 t) \]

we find in lowest order

\[ \omega_1 \simeq \omega_0 \]

and

\[ \delta \simeq \frac{1}{6} \beta A \]

\[ < x > = A_0 \simeq -\frac{1}{2} \beta A^2 \]
Please do not write on the back of these pages, use the front! There are five questions, each worth 20 points for a total of 100 points. You should first read all the problems, and then do them in an order which maximizes your point total. You must show how you get your answers. You will not get credit if you just write down an answer, unless it is obvious how you wrote it down, or you justify writing something down. However, you may write down the general solution to the equation of motion unless you are asked specifically to integrate it.

1) Consider a mass suspended between two springs as sketched below (no gravity). Only consider transverse motion. Each spring has an unstretched length of \( \ell \). Using the variables in the sketch below, \((y, h, d)\) find the kinetic and potential energy. You may expand the radical and keep the lowest order term, but do not expand until after you have calculated the potential energy. Write the Lagrangian and show that to lowest order the equation of motion is

\[
m\ddot{y} + 2k (d - \ell) \frac{y}{d} = 0.
\]

Now go back and calculate \(U(y)\) in next order, i.e. keep the next term in the binomial expansion. Is this potential symmetric or asymmetric about the equilibrium point? Which harmonics will participate in the motion? (explain your answer).
2) Consider the coupled system shown below. Three equal masses are connected by identical springs, and are constrained to move along a circle of radius $a$. (Ignore gravity in this problem.) Find the equations of motion, (any way you want to), find the eigenfrequencies, and describe the motion of the system for each of the modes. The equilibrium configuration is when the masses are $120^\circ$ apart.
3) A spool of thread (mass \( m \)) is allowed to unwind as shown below in the figure. Find \( T, U, \) and \( L \). Find the equation of constraint \( f(a, y, \phi) = 0 \). Now using the method of Lagrange multipliers find the equations of motion. Find the generalized forces of constraint, \( Q_y \) and \( Q_\phi \) and explain what they mean physically. \( I_{cm} = \frac{1}{2}ma^2 \). (You may assume the string is massless, and that all motion takes place in a plane.)
4) Consider the quadratic damping force, 

\[ F_d = -cv^2 \]

Consider a mass, with initial speed downward \( v_0 \), falling down in a viscous fluid which provides such a quadratic damping force. Assume a uniform gravitational field \( g \). Find an expression for the terminal velocity. Find the equation of motion and solve for \( v(t) \). Now find \( v(x) \). Assume the initial conditions: at \( t = 0, y = 0, v(0) = v_0 \). (Hint: there are new integrals on the front page.)
5) What are the fundamental differences between Lagrangian and Newtonian mechanics. What are the strong points of each? 

b) What is phase space and Liouville’s theorem? 

c) What is the definition of the Hamiltonian? 

d) What fundamental symmetries give the conservation of energy, momentum, and angular momentum?