NO BOOKS or NOTES ARE PERMITTED

\[ A_{xx}(\omega) = \frac{F_3}{\omega^2 - \omega_0^2 + \gamma^2 \omega^2} \]
\[ \tan \phi_D = \frac{\gamma \omega}{\omega^2 - \omega_0^2} \]
\[ \frac{1}{2} \pi L \bar{I}^2 \]

\[ \frac{1}{2} \frac{\pi^2}{c^2}, \qquad 1 + \tan^2 x = \sec^2 x \] where \( 1 \cos x = \sec x \)

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \]
\[ \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \]
\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \]
\[ \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \]
\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \]
\[ \cos 2x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x \]

Binomial: \( (1 + x)^n \simeq 1 + nx + \frac{n(n - 1)x^2}{2!} + \sum_{j=4}^{\infty} c_j x^{(j-1)} \) \( |x| < 1 \); \( c_j = \frac{n(n - 1)(n - 2) \ldots (n - (j - 2))}{(j - 1)!} \)

Taylor: \( f(x) = f(a) + (x - a)f'(a) + \frac{1}{2!}(x - a)^2 f''(a) + \ldots + \frac{1}{n!}(x - a)^n f^n(a) \)

Fourier: \( \psi(t) = A_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0t) \) \( A_0 = \frac{1}{\tau} \int_{0}^{\tau} \psi(t) \, dt \) \( \tau \Rightarrow 1 \) period

\[ a_m = \frac{2}{\tau} \int_{0}^{\tau} \psi(t) \cos(m\omega_0t) \, dt; \quad b_m = \frac{2}{\tau} \int_{0}^{\tau} \psi(t) \sin(m\omega_0t) \, dt; \quad \psi(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{im\omega_0t} \quad c_0 = A_0 \]

\[ c_n = \frac{1}{\tau} \int_{0}^{\tau} \psi(t) e^{-im\omega_0t} \, dt; \quad \psi(t) = \int_{-\infty}^{+\infty} C(\omega) e^{i\omega t} \, d\omega; \quad C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \psi(t) e^{-i\omega t} \, dt \]

\[ \int x \cos ax \, dx = \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax; \quad \int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax \]

\[ \int \ln(ax + b) \, dx = \frac{ax + b}{a} \ln(ax + b) - x \]

\[ \int \frac{dv}{(1 - v^2)^{3/2}} = c \tanh^{-1} \frac{v}{c} \] where \( \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} \]

\[ 1 \]
\[ 2 \]
\[ 3 \]
\[ 4 \]
\[ 5 \]

\[ \text{total} \]
For the simplest nonlinear oscillator, there are two possible cases:

i). The force is symmetric about the equilibrium point. If the force is represented by

\[ F = -(1 + \alpha x^2) \, sx \]

then the simplest form of the solution is

\[ x = A (\cos \omega_1 t + \epsilon \cos 3\omega_1 t) \]

we find in lowest order

\[ \omega_1 \simeq \omega_0 \left(1 + \frac{3}{8}\alpha A^2\right) \]

and

\[ \epsilon \simeq \frac{1}{32}\alpha A^2 \]

ii). The force is not symmetric about the equilibrium point and the simplest form for the force is

\[ F = -(1 + \beta x) \, sx \]

and the simplest form of the solution is

\[ x = A_0 + A (\cos \omega_1 t + \delta \cos 2\omega_1 t) \]

we find in lowest order

\[ \omega_1 \simeq \omega_0 \]

and

\[ \delta \simeq \frac{1}{6}\beta A \]

\[ <x> = A_0 \simeq -\frac{1}{2}\beta A^2 \]
Please do not write on the back of these pages, use the front! There are five questions, each worth 20 points for a total of 100 points. You should first read all the problems, and then do them in an order which maximizes your point total. You must show how you get your answers. You will not get credit if you just write down an answer, unless it is obvious how you wrote it down, or you justify writing something down. However, you may write down the general solution to the equation of motion unless you are asked specifically to integrate it.

1) A mass is suspended between two springs which have been stretched from their unstretched length by a large amount. Assume that the tension, \( T \), in the spring does not change by much when it is pulled transversely. Find the equation of motion for transverse motion. Make the appropriate expansion and keep the linear plus the first non-linear term. Is this force harder or softer than linear? Which harmonics will participate in the motion. Briefly tell why. For an amplitude \( A_0 \), find the amount of third harmonic for this system.
2) Consider the underdamped harmonic oscillator. Find the specific solution for the initial conditions:

\[ x(0) = A_0, \quad \text{and} \quad \dot{x}(0) = 0. \]
3) Two masses, each of value $m$ are connected to identical springs of spring constant $k$. Each mass is subject to a damping force $F = -b_1 \dot{x}$. One of the masses slides on top of the other and a viscous damping force, proportional to the relative velocity between the two masses, acts on them with a proportionality constant $b_2$

i. Find the eigenfrequencies, and the eigenwidths for this system.

ii. Sketch the mode configurations.

iii. What are the eigenvectors and mode coordinates for this motion?

iv. What is the electrical analogue of this system? (sketch the circuit)
4) A boat moves with initial speed $v_0$ on a lake. The viscous drag force of the lake on the boat is given by $F = -\alpha v$. At $t = 0$ the force which is propelling the boat forward is stopped, so that only the drag force is present.
   i. Find an expression for the speed $v(t)$.
   ii. Find the time for the boat to stop.
   iii. Find the distance the boat travels.
5)  
   a) Give three examples of resonance. Clearly identify the driving force, the response and the frequency dependence.
   b) Why are automobiles underdamped rather than critically damped or overdamped?
   c) Make a table of the analogous quantities for a mass on a spring and a simple \( LCR \) circuit. Include the differential equation for each case.
   d) Tell why a car would have trouble driving along on a bumpy road sketched below at one speed, but would have much less trouble above or below that speed.