NO BOOKS or NOTES ARE PERMITTED

\[ A(x) = \frac{F_0}{[\omega_0^2 - \omega^2 + \gamma^2 \omega^2]^\frac{1}{2}} \quad \tan \phi_D = \frac{-\gamma \omega}{\omega_0^2 - \omega^2} \]

\[ \frac{1}{2} L \omega^2, \quad \frac{1}{2} \frac{q^2}{c}, \quad 1 + \tan^2 x = \sec^2 x \text{ where } 1/ \cos x = \sec x \]

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \]

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \]

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \quad \cos 2x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x \]

**Binomial:** \( (1 + x)^n \approx 1 + nx + \frac{n(n-1)x^2}{2!} + \sum_{j=4}^{\infty} c_j x^{(j-1)} \quad |x| < 1; \quad c_j = \frac{n(n-1)(n-2) \cdots (n-(j-2))}{(j-1)!} \]

**Taylor:** \( f(x) = f(a) + (x - a)f'(a) + \frac{1}{2!}(x - a)^2 f''(a) + \ldots \]

**Fourier:** \( \psi(t) = A_0 + \sum_{n=1}^{\infty} a_n \cos n \omega t + \sum_{n=1}^{\infty} b_n \sin n \omega t \quad A_0 = \frac{1}{\tau} \int_{0}^{\tau} \psi(t) \, dt \quad \tau \Rightarrow 1 \text{ period} \)

\[ a_n = \frac{2}{\tau} \int_{\tau} \psi(t) \cos n \omega t \, dt; \quad b_n = \frac{2}{\tau} \int_{\tau} \psi(t) \sin n \omega t \, dt; \quad \psi(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{i n \omega t} \quad c_0 = A_0 \]

\[ c_n = \frac{1}{\tau} \int_{\tau} \psi(t) e^{-i n \omega t} \, dt; \quad \psi(t) = \int_{-\infty}^{+\infty} C(\omega) e^{i \omega t} \, d\omega; \quad C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \psi(t) e^{-i \omega t} \, dt \]

\[ \int x \cos ax \, dx = \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax; \quad \int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax \]

\[ \int \ln(a x + b) \, dx = \frac{ax + b}{a} \ln(ax + b) - x \]

\[ \int \frac{dv}{(1 - v^2)^{\frac{3}{2}}} = c \tanh^{-1} \frac{v}{c} \quad \text{where} \quad \tanh x = \frac{\sinh x}{\cosh x} \quad \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1} \]

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For the simplest nonlinear oscillator, there are two possible cases:

i). The force is symmetric about the equilibrium point. If the force is represented by

\[ F = - \left( 1 + \alpha x^2 \right) sx \]

then the simplest form of the solution is

\[ x = A (\cos \omega_1 t + \epsilon \cos 3\omega_1 t) \]

we find in lowest order

\[ \omega_1 \simeq \omega_0 \left( 1 + \frac{3}{8} \alpha A^2 \right) \]

and

\[ \epsilon \simeq \frac{1}{32} \alpha A^2 \]

ii). The force is not symmetric about the equilibrium point and the simplest form for the force is

\[ F = -(1 + \beta x) sx \]

and the simplest form of the solution is

\[ x = A_0 + A (\cos \omega_1 t + \delta \cos 2\omega_1 t) \]

we find in lowest order

\[ \omega_1 \simeq \omega_0 \]

and

\[ \delta \simeq \frac{1}{6} \beta A \]

\[ < x >= A_0 \simeq -\frac{1}{2} \beta A^2 \]
Please do not write on the back of these pages, use the front! There are five questions, each worth 20 points for a total of 100 points. You should first read all the problems, and then do them in an order which maximizes your point total. You must show how you get your answers. You will not get credit if you just write down an answer, unless it is obvious how you wrote it down, or you justify writing something down. However, you may write down the general solution to the equation of motion unless you are asked specifically to integrate it.

1) Two masses, \( m \) and \( 2m \) are suspended between three equal springs which have been stretched from their unstretched length by a large amount, such that the equilibrium separation of the masses is \( a \). Assume that the tension, \( T \), in the spring does not change by much when it is pulled transversely.
   1a. Find the equations of motion for transverse motion.
   1b. Find the eigenfrequencies, the eigenvectors, and the mode coordinates.
   1c. Sketch the mode configurations.
   1d. If a transverse harmonic driving force is applied to the right hand support, which modes will be excited? If this same driving force is applied to the middle, which modes will be excited? Please briefly explain these answers.
2) The restoring force in problem 1 for transverse oscillations is nonlinear. Please write down the force, keeping the lowest order nonlinear terms, and find the coupled (non-linear equations of motion). Is this non-linear restoring force symmetric or asymmetric? Which frequencies do you expect to be present in the motion for moderate amplitudes?
3) A simple pendulum has a length $\ell$, and has a velocity dependent damping force. A driving force is applied to the pivot point, which moves a distance $\zeta$. Take the coordinate system $x = 0$ to be below the point $\zeta = 0$ as shown in the attached figure.

a. Show that for small oscillations the equation of motion is

$$\ddot{x} + \gamma \dot{x} + \frac{g}{\ell} x = \frac{g}{\ell} \zeta$$

b. What is the steady state solution for this motion if $\zeta = \zeta_0 \cos \omega t$?

c. Find the amplitude at $\omega = \omega_0$ resonance.

e. Find an expression for $\gamma$ if the amplitude is halved in 100 cycles.
4) A block of mass $m$ slides down a frictionless incline. The block is released from a height $h$ above the bottom of the loop.
4a. What is the force of the inclined track on the block at point A?
4b. What is the force of the track on the block at point B?
4c. At what speed does the Block leave the track?
4d. How far away from point A, does the block land on level ground?
5) Consider a positively charged particle moving in a region which can contain both magnetic and electric fields.
   a) Write down the (Lorentz) force which this particle experiences.
   b) If there is no electric field, and the velocity is transverse to a uniform magnetic field, show that the trajectory is a circle. Find an expression for the radius, and find the cyclotron frequency (the frequency of revolution).
   c) Suppose that
      \[ \vec{B} = B \hat{k}, \quad \vec{E} = E_y \hat{j} + E_z \hat{k} \]
      Show that the z component of motion is given by
      \[ z(t) = z_0 + \dot{z}_0 + \frac{qE_z}{2m} t^2 \]
      where \( z(0) \equiv z_0 \) and \( \dot{z}(0) \equiv \dot{z}_0 \).
   d) Re-write the equations of motion in terms of \( v_x \) and \( v_y \), and the appropriate time derivatives. Show that the velocities satisfy harmonic oscillator-like equations, and obtain expressions for \( \dot{x}(t) \) and \( \dot{y}(t) \). Show that the time averages of these velocity components is
      \[ < \dot{x} > = \frac{E_y}{B}; \quad < \dot{y} > = 0 \]