PY408 Mid-term Examination
October 31, 2002

NO BOOKS or NOTES ARE PERMITTED

\[ A_{z\xi}(\omega) = \frac{F_{z\xi}}{m} \left( \frac{\omega}{\omega_0^2 - \omega^2} \right)^{\frac{1}{2}} \]

\[ \tan \phi_D = \frac{\omega}{\omega_0^2 - \omega^2} \]

\[ \frac{1}{2} L^2, \quad \frac{1}{2} m \frac{d^2}{dt^2} x = \text{sec}^2 x \text{ where } 1/\cos x = \sec x \quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \quad \sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \]

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \quad \cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \]

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \quad \cos 2x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x \]

Binomial: \( (1 + x)^n \approx 1 + nx + \frac{n(n-1)x^2}{2!} + \sum_{j=1}^{\infty} c_j x^{j-1} \quad |x| < 1; \quad c_j = \frac{n(n-1) \cdots (n-j+1)}{(j-1)!} \)

Taylor: \( f(x) = f(a) + (x - a) f'(a) + \frac{1}{2!}(x - a)^2 f''(a) + \ldots + \frac{1}{n!}(x - a)^n f^n(a) \)

Fourier: \( \psi(t) = A_0 + \sum_{n=1}^{\infty} a_n \cos(n \omega_1 t) + \sum_{n=1}^{\infty} b_n \sin(n \omega_1 t) \quad A_0 = \frac{1}{\tau} \int_0^{\tau} \psi(t) \, dt \quad \tau \Rightarrow 1 \text{ period} \)

\[ a_n = \frac{2}{\tau} \int_{\tau} \psi(t) \cos(m \omega_1 t) \, dt; \quad b_n = \frac{2}{\tau} \int_{\tau} \psi(t) \sin(m \omega_1 t) \, dt; \quad \psi(t) = \sum_{n=-\infty}^{\infty} c_n e^{i \omega_1 t} \quad c_0 = A_0 \]

\[ c_n = \frac{1}{\tau} \int_{\tau} \psi(t) \, e^{-i \omega_1 t} \, dt; \quad \psi(t) = \int_{-\infty}^{+\infty} C(\omega) \, e^{i \omega t} \, d\omega; \quad C(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \psi(t) \, e^{-i \omega t} \, dt \]

\[ \int \cos ax \, dx = \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax; \quad \int \sin ax \, dx = \frac{1}{a^2} \sin ax + \frac{x}{a} \cos ax \]

\[ \int \ln(ax + b) \, dx = \frac{a x + b}{a} \ln(ax + b) - x \quad \int \sin^2(ax) \, dx = \frac{x}{2} - \frac{1}{4a} \sin(2ax) \]

\[ \int \frac{dv}{(1 - v^2)^{3/2}} = c \tanh^{-1} \frac{v}{c} \quad \text{where} \quad \tan h x = \frac{\sinh x}{c \cosh x} \quad e^x - e^{-x} \quad e^x + e^{-x} = \frac{e^{2x} - 1}{e^{2x} + 1} \]

\[ 1 \]

\[ 2 \]

\[ 3 \]

\[ 4 \]

total
For the simplest nonlinear oscillator, there are two possible cases:

i). The force is symmetric about the equilibrium point. If the force is represented by

\[ F = - \left( 1 + \alpha x^2 \right) s x \]

then the simplest form of the solution is

\[ x = A (\cos \omega_1 t + \epsilon \cos 3\omega_1 t) \]

we find in lowest order

\[ \omega_1 \simeq \omega_0 \left( 1 + \frac{3}{8} \alpha A^2 \right) \]

and

\[ \epsilon \simeq \frac{1}{32} \alpha A^2 \]

ii). The force is not symmetric about the equilibrium point and the simplest form for the force is

\[ F = - (1 + \beta x) s x \]

and the simplest form of the solution is

\[ x = A_0 + A (\cos \omega_1 t + \delta \cos 2\omega_1 t) \]

we find in lowest order

\[ \omega_1 \simeq \omega_0 \]

and

\[ \delta \simeq \frac{1}{6} \beta A \]

\[ < x > = A_0 \simeq -\frac{1}{2} \beta A^2 \]
Please do not write on the back of these pages, use the front! There are four questions, each worth 25 points for a total of 100 points. You should first read all the problems, and then do them in an order which maximizes your point total. You must show how you get your answers. You will not get credit if you just write down an answer, unless it is obvious how you wrote it down, or you justify writing something down. However, you may write down the general solution to the equation of motion unless you are asked specifically to integrate it.

1) (5,5,5,5) Two equal masses are suspended between three springs as shown below. The middle spring has a different spring constant from the outer ones. Each mass is subject to a velocity dependent damping force, \( f_d = -b\dot{z} \). The springs are pre-stretched to a tension \( T \). For longitudinal motion,
   i.) find the eigenfrequencies.
   ii.) find the eigenvectors. (hint: are the eigenvectors changed by the presence of damping?) Once you obtain the relationship between \( A_1 \) and \( A_2 \) you may just write down the eigenvectors.
   iii.) find the mode coordinates.
   iv.) sketch the mode configurations.
   v.) suppose one mass is held fixed, and the other is pulled to the side some distance. Both are released from rest at \( t = 0 \). Describe the subsequent motion. (This behavior is especially obvious if the coupling spring is weak.)
2) (5,10,5,5) Now consider the system from problem 1, with a driving force \( f(t) = f_0 \cos \omega t \) attached at the mid-point of the middle spring (which is indicated as a two-headed arrow on the sketch). Thus the center of the spring is pulled back and forth harmonically, oscillating in the two directions shown by the arrows. You may neglect damping in this question, except when sketching part iv.

i.) Write down the equations of motion (assuming that the driving force is transmitted through the spring unchanged).

ii.) Use the mode coordinates to uncouple the equations.

iii. Which modes are excited by this driving force? Tell why you might have excited this result.

iv.) Sketch the amplitude of mode 1 and mode 2 vs. driving frequency for this specific case. You may assume that the mode widths are much narrower than the difference between the mode frequencies. What difference would it make if they weren't?
3) (6,6,7=4+3) Consider the following two velocity dependent damping forces:

\[ f_{1d} = -c_1 v; \quad f_{2d} = -c_2 v^2 \]

Consider a system which at \( t = 0 \) has a mass starting from \( x = 0 \), travelling in a line with speed \( v = v_0 \). No other force except the damping force acts on the system. For each of these forces:

i.) Find an expression for the speed \( v(t) \).

ii.) Find an expression for \( v(x) \).

iii.) Find an expression for \( x(t) \).

iv.) Find the total distance travelled in each case. Explain the difference you find for the total distance travelled between the quadratic force and the linear one. Is your answer consistent with what you would expect from part iii (check it out, i.e. don’t just say yes or no)?
4) (5,5,5,10)
   i.) Two frequencies $f_1$ and $f_2$ act on a nonlinear system. Give the resulting frequencies up to second order.
   ii.) Now suppose that $f_2 \approx 2f_1$, say $f_2 = 2f_1 + \delta$. What are the resulting frequencies up to second order. (This effect helps musicians tune intervals, since as $\delta \to 0$ the sound “clears” and sounds better.)
   iii.) Explain the statement: “The expansion of solids is a direct result of the asymmetry of the interatomic potential. Use a sketch and physical arguments why this is so.
   iv.) Consider a mass placed 1/3 of the way along a string which has tension $T$. Assume that the tension does not change significantly when the mass is displaced transversely. Find the restoring force for a transverse displacement, and expand it to second order (keep 2 terms). What harmonics will be present in the motion (and explain briefly why you say what you say).